1 Vector Calculus

We focus in this section on the 3-dimensional vector space.

We can write vectors using orthogonal unit vectors along the coordinate axes, e.g.

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

You should be very familiar with the basic vector algebra such as addition, magnitude, etc. The dot and vector products are defined as follows:

- dot/inner product:
$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$
where
$$\begin{aligned} \mathbf{a} &= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \\ \mathbf{b} &= b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \end{aligned}$$
and it can be shown that
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b})$$

- cross/vector product:
$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$
$\mathbf{v}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$, with its direction defined by the right-hand rule. Geometrically, $|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\sin\angle(\mathbf{a},\mathbf{b})|$ is the area of the parallelogram constructed by $\mathbf{a}$ and $\mathbf{b}$.

In more details, the determinant form of the vector product is obtained as follows. Based on the right-hand rule and the definition of cross products, we have

$$\begin{aligned} \mathbf{i}\times\mathbf{i} = 0,\ \mathbf{i}\times\mathbf{j} = \mathbf{k},\ \mathbf{i}\times\mathbf{k} = -\mathbf{j} \\ \mathbf{j}\times\mathbf{i} = -\mathbf{k},\ \mathbf{j}\times\mathbf{j} = 0,\ \mathbf{j}\times\mathbf{k} = \mathbf{i} \\ \mathbf{k}\times\mathbf{i} = \mathbf{j},\ \mathbf{k}\times\mathbf{j} = -\mathbf{i},\ \mathbf{k}\times\mathbf{k} = 0 \end{aligned}$$

and hence

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_x b_y\mathbf{k} - a_x b_z\mathbf{j} - a_y b_x\mathbf{k} + a_y b_z\mathbf{i} + a_z b_x\mathbf{j} - a_z b_y\mathbf{i} \end{aligned}$$

which exactly equals the determinant formula.

To see the intuition of $|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\sin\angle(\mathbf{a},\mathbf{b})|$, let $\mathbf{a}$ be aligned with the $\mathbf{i}$ axis and $\mathbf{b}$ be decomposed as $|\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b})\,\mathbf{i} + |\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})\,\mathbf{j}$ (if not, we can rotate $\mathbf{a}$ and $\mathbf{b}$ by the same angle to achieve the alignment). Then

$$\begin{aligned} |\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| &= \left||\mathbf{a}|\,\mathbf{i} \times \left(|\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b})\,\mathbf{i} + |\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})\,\mathbf{j}\right)\right| \\ &= \left||\mathbf{a}|\,\mathbf{i} \times |\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})\,\mathbf{j}\right| \\ &= |\mathbf{a}|\,|\mathbf{b}|\,|\sin\angle(\mathbf{a},\mathbf{b})| \end{aligned}$$

Triple scalar product:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

is the volume of the parallelepiped constructed by **a**, **b**, and **c**.

Let f be a scalar function of x, y, and z. ∇f is the **gradient** vector

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

and we can write

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

from which we can define the gradient operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Divergence and Curl: if

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

then the divergence (a scalar) is defined by

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

and the curl by

$$\mathrm{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Exercise 1. Let **a**, **b**, and **c** be 3-dimensional vectors. Prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$