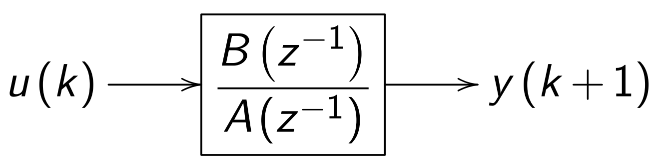
PAA with Parallel Predictors

Big picture: we know now...



simply means:

$$\begin{aligned} y(k+1) &= B\left(z^{-1}\right)u(k) - \left(A\left(z^{-1}\right)-1\right)y(k+1) \\ &= \theta^T \phi(k) \end{aligned}$$

In RLS:

$$\hat{y}^o(k+1) = \hat{\theta}^T(k)\phi(k) = \hat{B}\left(z^{-1},k\right)u(k) - \left(\hat{A}\left(z^{-1},k\right)-1\right)y(k+1)$$

Understanding the notation: if $B\left(z^{-1}\right) = b_o + b_1 z^{-1} + \cdots + b_m z^{-m}$,
then $\hat{B}\left(z^{-1},k\right) = \hat{b}_o(k) + \hat{b}_1(k)z^{-1} + \cdots + \hat{b}_m(k)z^{-m}$
Remark: z^{-1}–shift operator; some references use q^{-1} instead

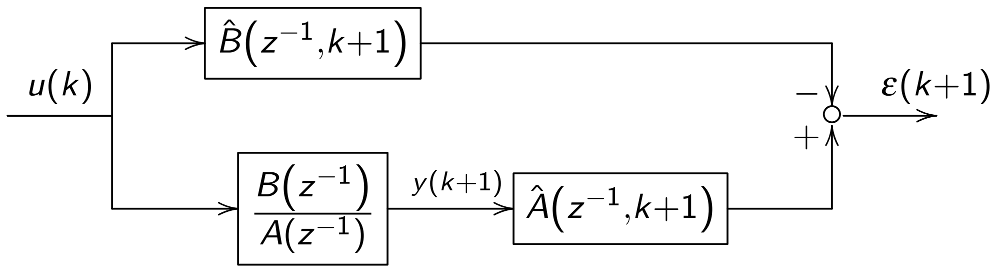
RLS is a series-parallel adjustable system

RLS in *a posteriori* form:

$$\hat{y}(k+1) = \hat{B}\left(z^{-1}, k+1\right) u(k) - \left(\hat{A}\left(z^{-1}, k+1\right) - 1\right) y(k+1)$$

prediction error:

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) = \hat{A}\left(z^{-1}, k+1\right) y(k+1) - \hat{B}\left(z^{-1}, k+1\right) u(k)$$



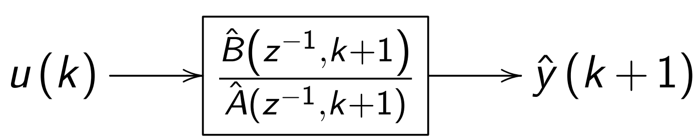
A series-parallel structure: $\hat{A}\left(z^{-1}, k+1\right)$–in series with plant; $\hat{B}\left(z^{-1}, k+1\right)$–in parallel with the plant

Observation

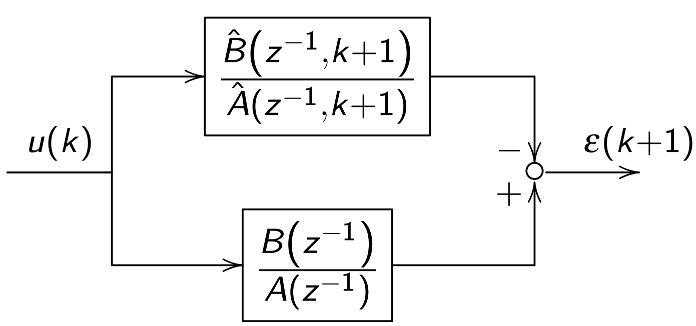
If hyperstability holds such that $\varepsilon(k+1) \to 0$, $\hat{y}(k+1) \to y(k+1)$, it seems fine to do instead:

$$\hat{y}(k+1) = \hat{B}\left(z^{-1}, k+1\right) u(k) - \left(\hat{A}\left(z^{-1}, k+1\right) - 1\right) \hat{y}(k+1) \quad (1)$$

i.e.

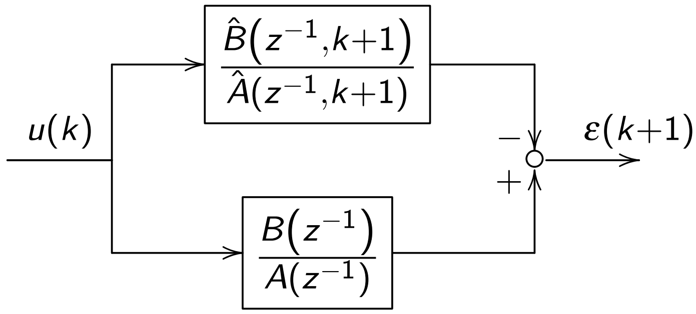


then we have a parallel structure

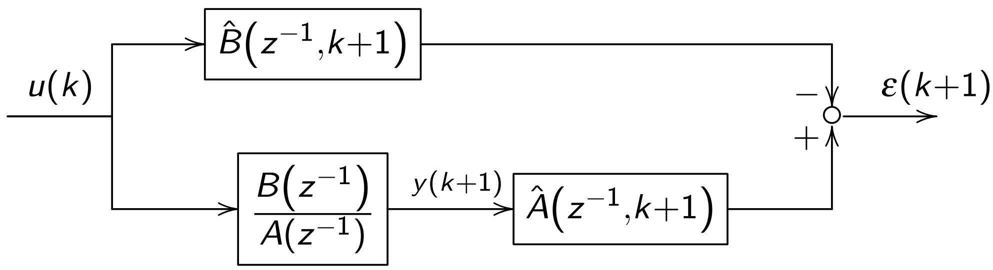


- it turns out this brings certain advantages

Other names



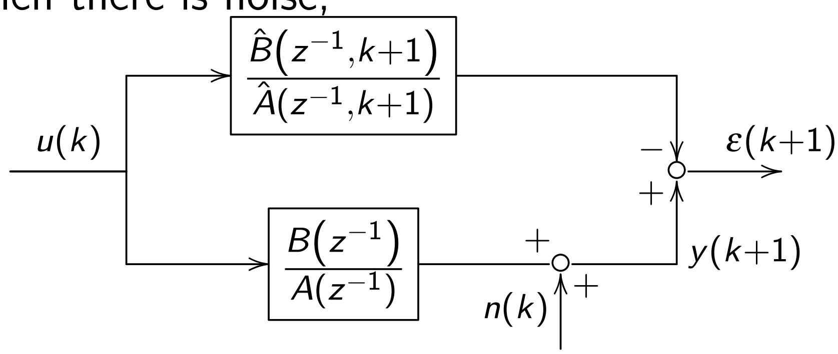
is also called an output-error method



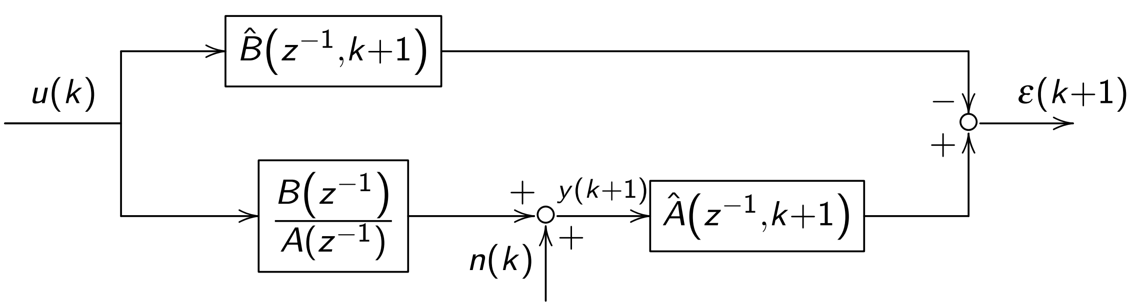
is also called an equation-error method

Benefits of parallel algorithms

Intuition: when there is noise,



provides better convergence of $\hat{\theta}$ than



We will talk about the PAA convergence in a few more lectures.

Outline

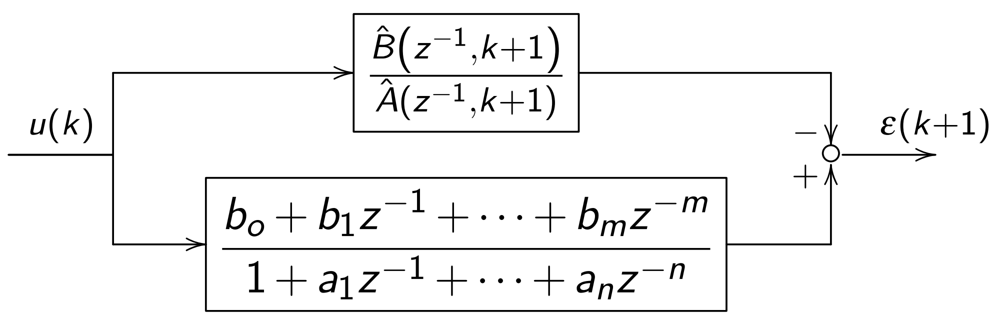
1. Big picture
   Series-parallel adjustable system (equation-error method)
   Parallel adjustable system (output-error method)

2. RLS-based parallel PAA
   Formulas
   Stability requirement for PAAs with fixed adaptation gain
   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)

RLS based parallel PAA



PAA summary:

- *a priori* $$\hat{\theta}(k+1)=\hat{\theta}(k)+\frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$
- *a posteriori* $$\hat{\theta}(k+1)=\hat{\theta}(k)+F(k)\phi(k)\varepsilon(k+1)$$

$$F^{-1}(k+1)=\lambda_1(k)F^{-1}(k)+\lambda_2(k)\phi(k)\phi^T(k)$$

$$\boxed{\phi^T(k)=[-\hat{y}(k),-\hat{y}(k-1),\ldots,-\hat{y}(k+1-n),u(k),\ldots,u(k-m)]}$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

parameter estimation error :

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

a posteriori prediction error : $y(k+1) = \frac{B(z^{-1})}{A(z^{-1})}u(k)$ gives

$$B\left(z^{-1}\right)u(k) = A\left(z^{-1}\right)y(k+1)$$
$$\hat{B}\left(z^{-1},k+1\right)u(k) = \hat{A}\left(z^{-1},k+1\right)\hat{y}(k+1)$$

hence

$$A\left(z^{-1}\right)y(k+1) - \hat{A}\left(z^{-1},k+1\right)\hat{y}(k+1)\boxed{\pm A\left(z^{-1}\right)\hat{y}(k+1)}$$
$$= B\left(z^{-1}\right)u(k) - \hat{B}\left(z^{-1},k+1\right)u(k)$$

i.e. $A\left(z^{-1}\right)\varepsilon(k+1) = [B\left(z^{-1}\right) - \hat{B}\left(z^{-1},k+1\right)]u(k)$
$-[A\left(z^{-1}\right) - \hat{A}\left(z^{-1},k+1\right)]\hat{y}(k+1)$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

a posteriori prediction error (cont'd):

$$A\left(z^{-1}\right)\varepsilon(k+1) = \overbrace{\left[B\left(z^{-1}\right) - \hat{B}\left(z^{-1},k+1\right)\right]u(k)}^{[\bigstar]}$$
$$-\left[A\left(z^{-1}\right) - \hat{A}\left(z^{-1},k+1\right)\right]\hat{y}(k+1)$$

Look at [★]: $B\left(z^{-1}\right) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}$ gives

$$\left[B\left(z^{-1}\right) - \hat{B}\left(z^{-1},k+1\right)\right]u(k)$$

$$= \begin{bmatrix} b_0 - \hat{b}_0(k+1) \\ b_1 - \hat{b}_1(k+1) \\ \vdots \\ b_m - \hat{b}_m(k+1) \end{bmatrix}^T \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-m} \end{bmatrix} u(k) = \begin{bmatrix} b_0 - \hat{b}_0(k+1) \\ b_1 - \hat{b}_1(k+1) \\ \vdots \\ b_m - \hat{b}_m(k+1) \end{bmatrix}^T \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-m) \end{bmatrix}$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

Similarly, for $A\left(z^{-1}\right)=1+a_1z^{-1}+\cdots+a_nz^{-n}$

$$\left[\hat{A}\left(z^{-1},k+1\right)-A\left(z^{-1}\right)\right]\hat{y}\left(k+1\right)=\begin{bmatrix} a_1-\hat{a}_1\left(k+1\right)\\ a_2-\hat{a}_2\left(k+1\right)\\ \vdots\\ a_n-\hat{a}_n\left(k+1\right)\end{bmatrix}^T\begin{bmatrix} -\hat{y}\left(k\right)\\ -\hat{y}\left(k-1\right)\\ \vdots\\ -\hat{y}\left(k+1-n\right)\end{bmatrix}$$

Recall: $$\theta^T=\left[a_1,a_2,\cdots a_n,b_0,b_1,\cdots,b_m\right]^T$$

$$\phi\left(k\right)=\left[-\hat{y}\left(k\right),-\hat{y}\left(k-1\right),\ldots,-\hat{y}\left(k+1-n\right),u\left(k\right),\ldots,u\left(k-m\right)\right]$$

hence

$$A\left(z^{-1}\right)\varepsilon\left(k+1\right)=\left[B\left(z^{-1}\right)-\hat{B}\left(z^{-1},k+1\right)\right]u\left(k\right)$$
$$-\left[A\left(z^{-1}\right)-\hat{A}\left(z^{-1},k+1\right)\right]\hat{y}\left(k+1\right)=\underline{-\tilde{\theta}^T\left(k+1\right)\phi\left(k\right)}$$

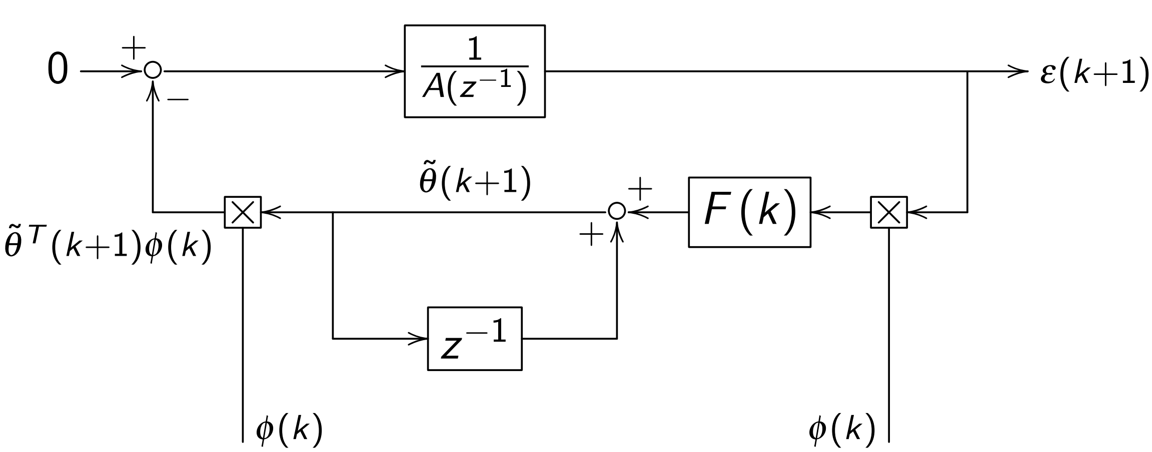
Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

PAA equations:

$$\tilde{\theta}\left(k+1\right)=\tilde{\theta}\left(k\right)+F\left(k\right)\phi\left(k\right)\varepsilon\left(k+1\right)$$
$$A\left(z^{-1}\right)\varepsilon\left(k+1\right)=-\tilde{\theta}^T\left(k+1\right)\phi\left(k\right)$$

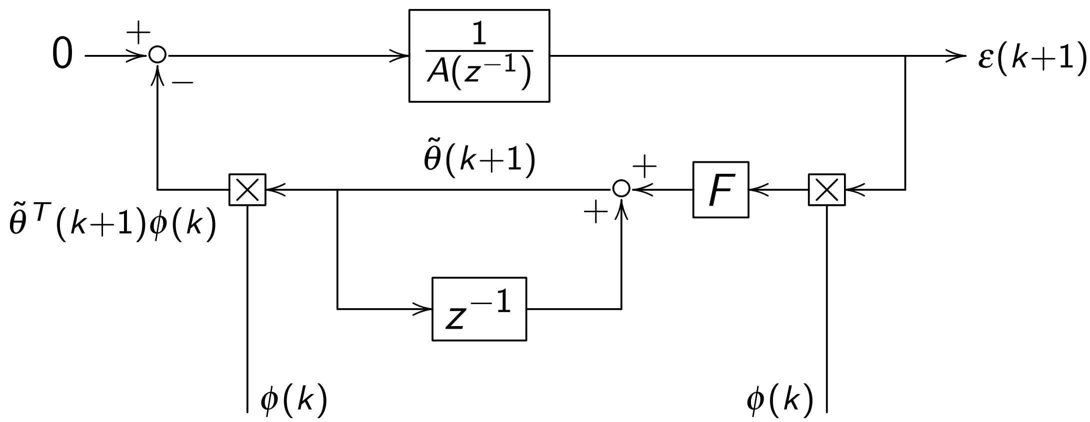
equivalent block diagram:



Stability of RLS based parallel PAA

step 2: Popov inequality

We will consider a simplified case with $F(k) = F \succ 0$:

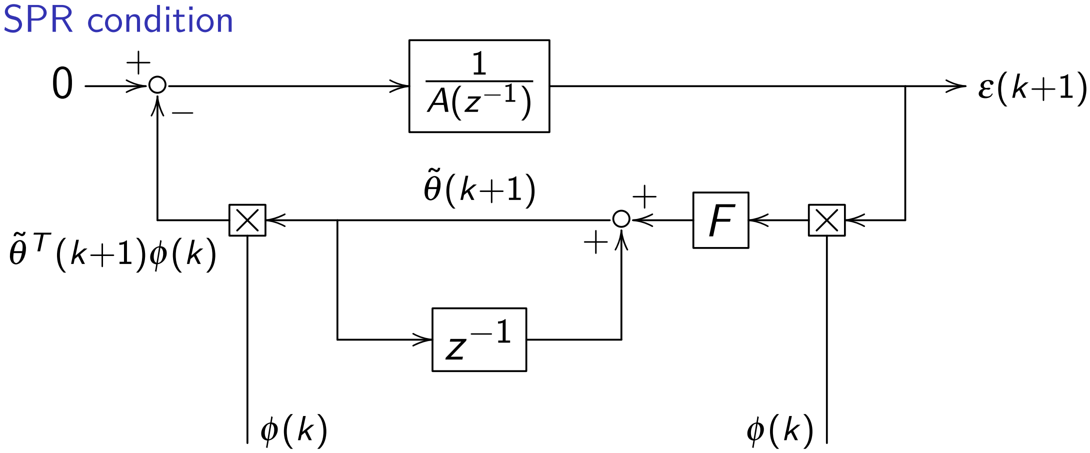


The nonlinear block is exactly the same as that in RLS, hence satisfying Popov inequality:

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$

Stability of RLS based parallel PAA

step 3: SPR condition



If $G(z^{-1}) = \frac{1}{A(z^{-1})}$ is SPR, then the PAA is asmptotically hyperstable

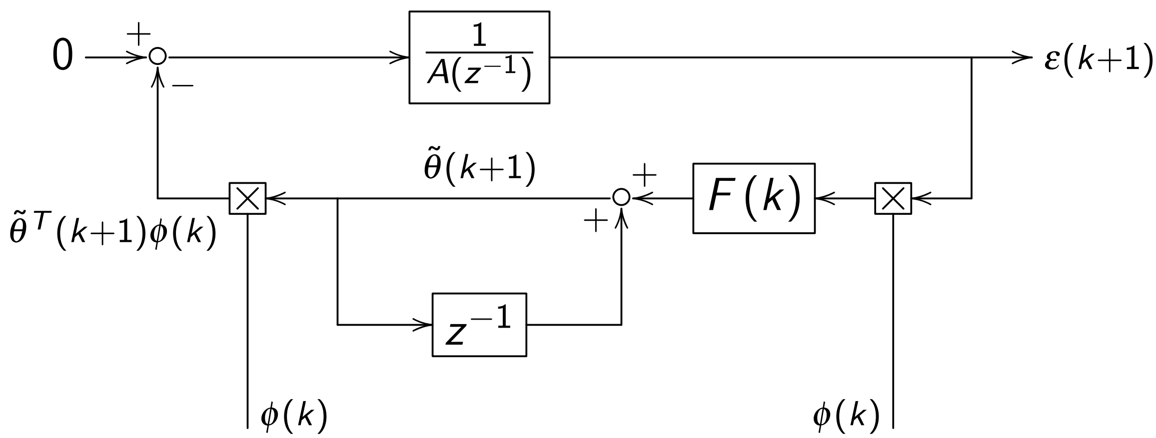
Remarks:

- RLS has an identity block: $G(z^{-1}) = 1$ which is independent of the plant
- $1/A(z^{-1})$ depends on the plant (usually not SPR)
- several other PAAs are developed to relax the SPR condition

Stability of RLS based parallel PAA: extension

For the case of a time-varying $F(k)$ with

$$F^{-1}(k+1) = \lambda_1(k) F^{-1}(k) + \lambda_2(k) \phi(k) \phi^T(k)$$



the nonlinear block is more involved; we'll prove later, that it requires

$$\frac{1}{A(z^{-1})} - \frac{1}{2}\lambda, \text{ where } \lambda = \max_k \lambda_2(k) < 2, \text{ to be SPR}$$

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Parallel algorithm with a fixed compensator

Instead of:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$

$$\phi^T(k) = [-\hat{y}(k), -\hat{y}(k-1), \ldots, -\hat{y}(k+1-n), u(k), \ldots, u(k-m)]$$

do: $$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\nu^o(k+1)$$

where

$$\nu(k+1) = C\left(z^{-1}\right)\varepsilon(k+1) = \left(c_0 + c_1 z^{-1} + \ldots c_n z^{-n}\right)\varepsilon(k+1)$$
$$\nu^o(k+1) = c_0\varepsilon^o(k+1) + c_1\varepsilon(k) + \ldots c_n\varepsilon(k-n+1)$$

Parallel algorithm with a fixed compensator

The SPR requirement becomes

$$\frac{C\left(z^{-1}\right)}{A(z^{-1})} - \frac{\lambda}{2},\ \lambda = \max_k \lambda_2(k) < 2 \qquad (2)$$

should be SPR.

Remark:

- if c_i's are close to a_i's, (2) approximates $1-\lambda/2 > 0$, and hence is likely to be SPR
- problem: $A\left(z^{-1}\right)$ is unknown *a priori* for the assigning of $C\left(z^{-1}\right)$
- solution: make $C\left(z^{-1}\right)$ to be adjustable as well

Parallel algorithm with an adjustable compensator

If $A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$, let $\hat{C}(z^{-1}) = 1 + \hat{c}_1 z^{-1} + \cdots + \hat{c}_n z^{-n}$ and

$$\nu(k+1) = \hat{C}\left(z^{-1}, k+1\right)\varepsilon(k+1)$$

$$\nu^o(k+1) = \varepsilon^o(k+1) + \sum_{i=1}^{n} \hat{c}_i(k)\varepsilon(k+1-i)$$

do

$$\hat{\theta}_e(k+1) = \hat{\theta}_e(k) + \frac{F_e(k)\phi_e(k)}{1+\phi_e^T(k)F_e(k)\phi_e(k)}\nu^o(k+1)$$

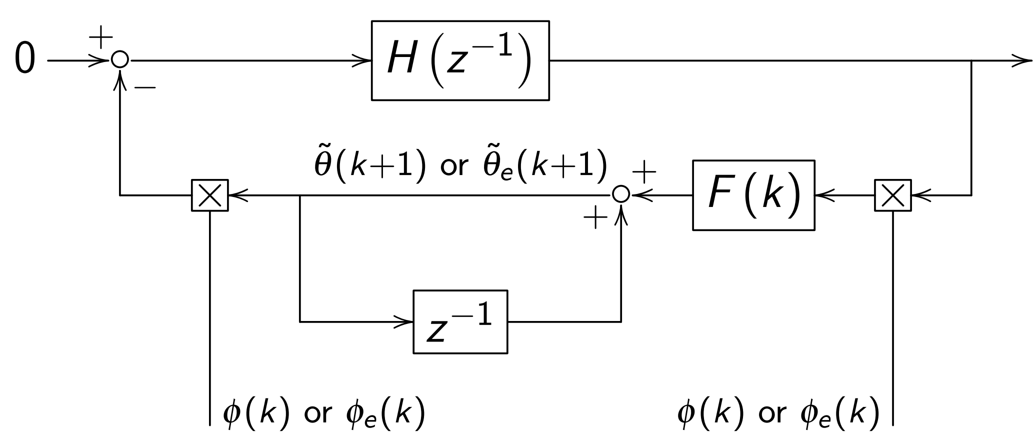
$$\hat{\theta}_e^T(k) = \left[\hat{\theta}^T(k), \hat{c}_1(k), \ldots, \hat{c}_n(k)\right]$$

$$\phi_e^T(k) = \left[\phi^T(k), -\varepsilon(k), \ldots, -\varepsilon(k+1-n)\right]$$

$$F_e^{-1}(k+1) = \lambda_1(k)F_e^{-1}(k) + \lambda_2(k)\phi_e(k)\phi_e^T(k)$$

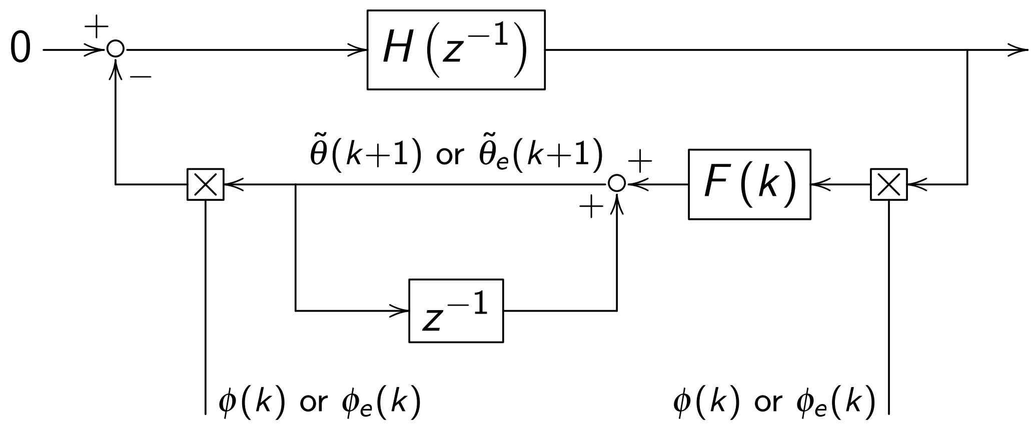
which has guaranteed **asymptotical stablility.**

General PAA block diagram



$H(z^{-1})$	PAA
1	RLS/parallel predictor with adjustable compensator
$1/A(z^{-1})$	parallel predictor
$C(z^{-1})/A(z^{-1})$	parallel predictor with fixed compensator

General PAA block diagram



- if $F(k) = F$, $H(z^{-1})$ being SPR is sufficient for asymptotic stability
- if $F(k)$ is time-varying, we will show next: $H(z^{-1}) - \frac{1}{2}\lambda$ being SPR is sufficient for asymptotic stability

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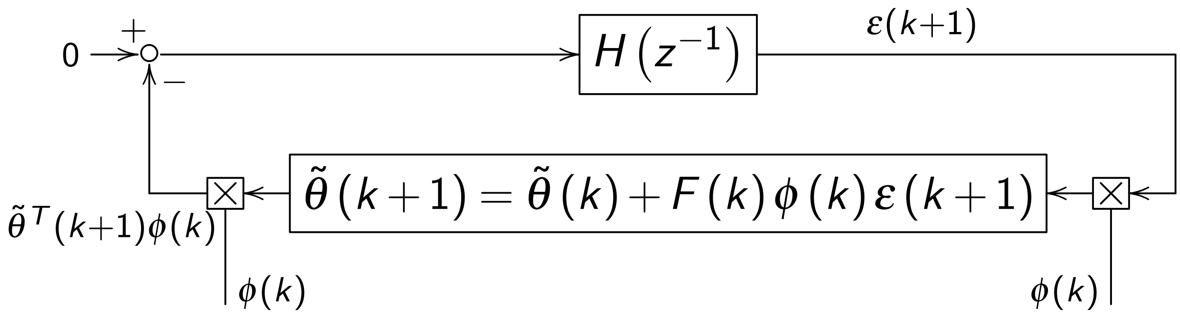
PAA with time-varying adaptation gains

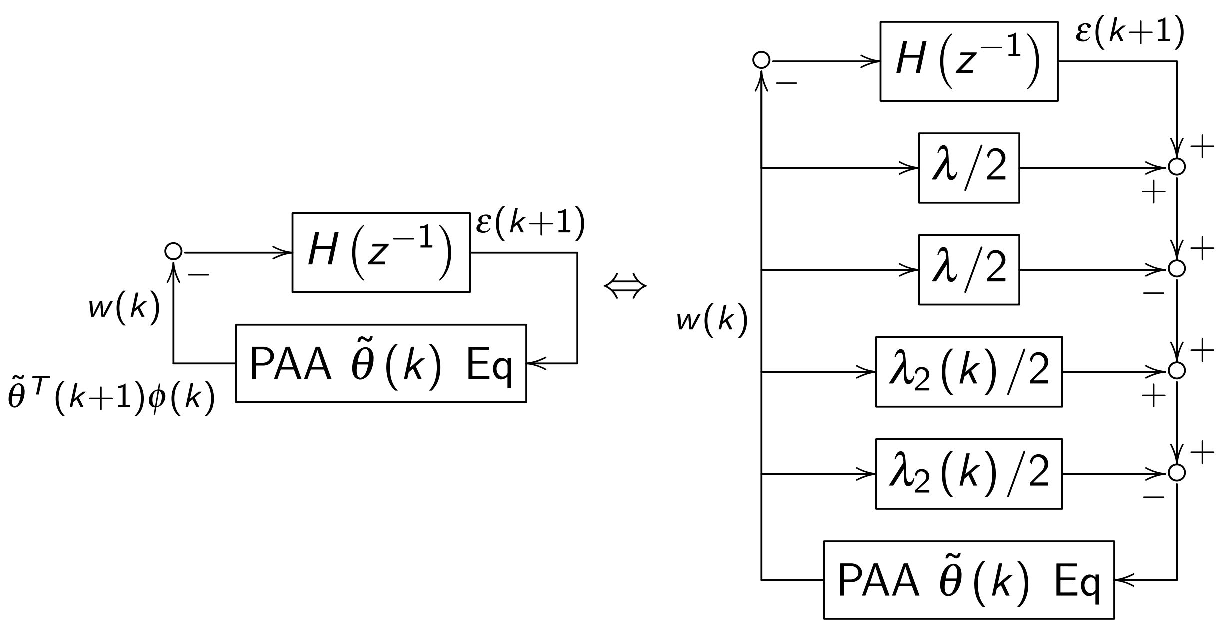


$$\text{where } F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi^T(k)\phi(k)$$

- unfortunately, the nonlinear block does not satisfy Popov inequality (not passive)

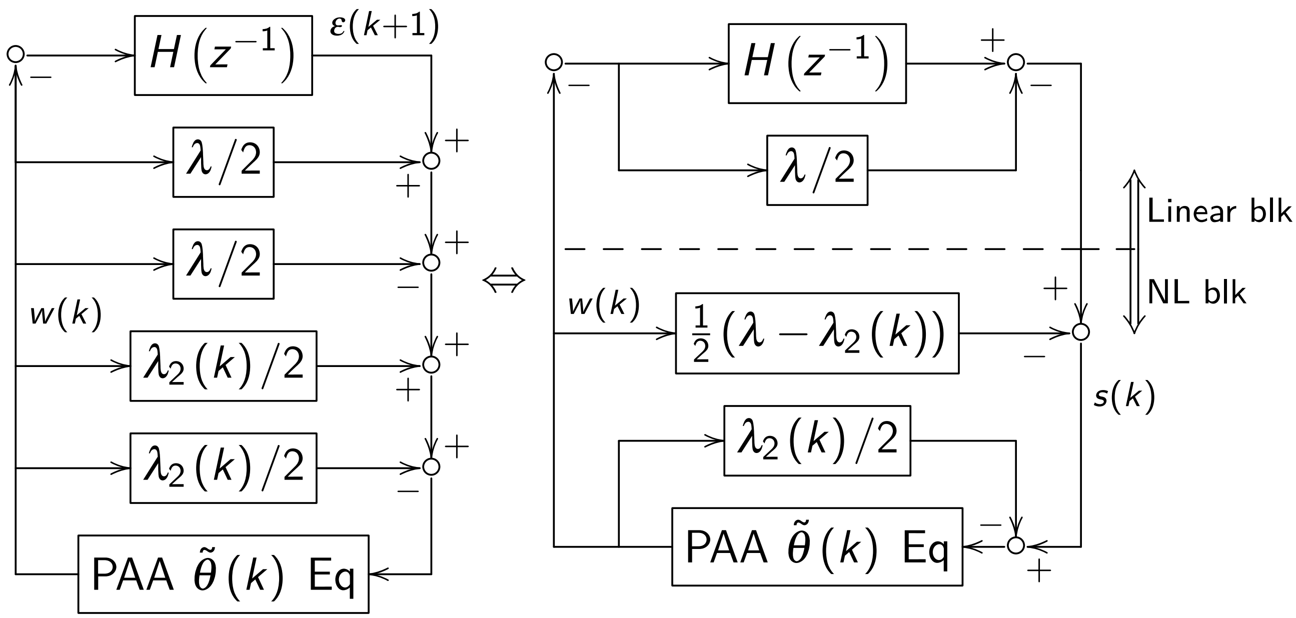
PAA with time-varying adaptation gains

a modification can re-gain the passivity of the feedback block



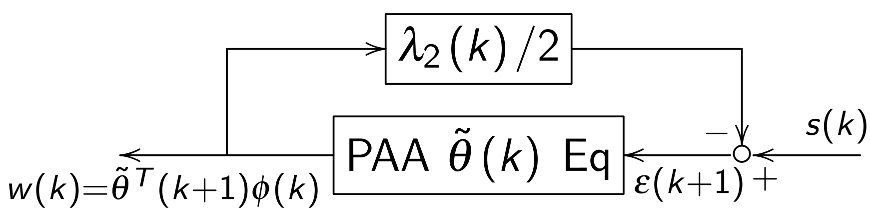
PAA with time-varying adaptation gains

a modification can re-gain the passivity of the feedback block

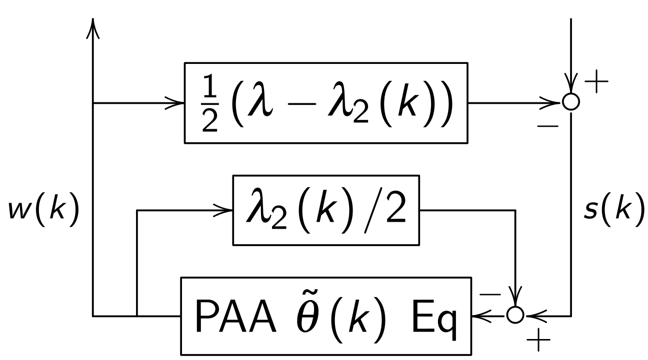


PAA with time-varying adaptation gains

step 1: show that the following is passive



step 2: the following is then passive

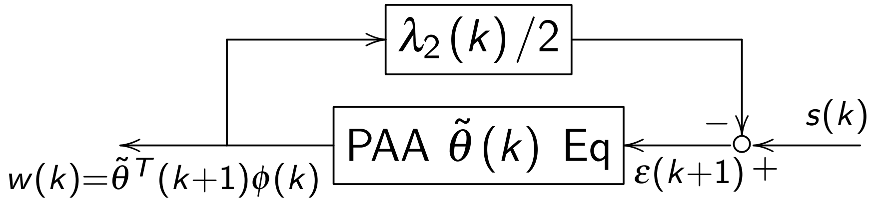


note that it is a feedback connection of a passive block with $\frac{1}{2}(\lambda-\lambda_2(k))\geq 0$

step 3: SPR condition for the linear block $H(z^{-1})-\frac{\lambda}{2}$

Passivity of the sub nonlinear block

Consider:



$s(k) = \varepsilon(k+1) + \frac{\lambda_2(k)}{2}\tilde{\theta}^T(k+1)\phi(k)$ gives

$$\sum_{k=0}^{k_1} w(k)s(k)$$

$$= \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\left[\varepsilon(k+1) + \frac{\lambda_2(k)}{2}\tilde{\theta}^T(k+1)\phi(k)\right]$$

$\Downarrow$note that $F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$

$$= \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) + \frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1) - \lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1)$$

which is no less than $-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)$ as shown next.

Proof of passivity of the sub nonlinear block

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

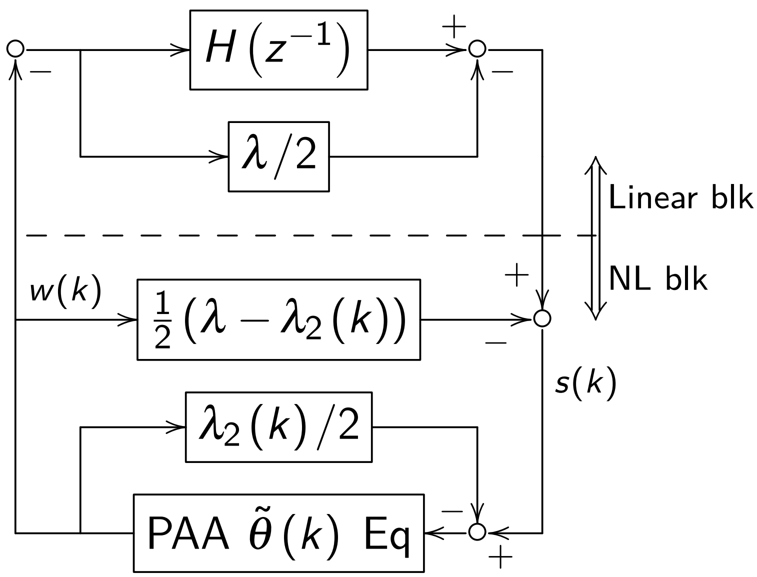
hence

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) = \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)F^{-1}(k)\left(\tilde{\theta}(k+1) - \tilde{\theta}(k)\right)$$

Combining terms and after some algebra (see appendix), we get

$$\sum_{k=0}^{k_1} w(k)s(k) = \sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)$$
$$+ \sum_{k=0}^{k_1} \frac{1}{2}\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]$$
$$+ \underbrace{\sum_{k=0}^{k_1} \frac{1}{2}\left[\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)\right]}_{\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}(k_1)\tilde{\theta}(k_1+1) - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)} \quad (3)$$

Summary



In summary, the NL block indeed satisfies Popov inequality.
For stability of PAA, it is sufficient that

$$H\left(z^{-1}\right)-\frac{\lambda}{2} \text{ is SPR}$$

Appendix: derivation of (3)

$$\begin{aligned}
&\sum_{k=0}^{k_1} \underline{\tilde{\theta}^T(k+1)F^{-1}(k)\left(\tilde{\theta}(k+1)-\tilde{\theta}(k)\right)}+\frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1)-\lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1)\\
&=\sum_{k=0}^{k_1} \underline{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)}+\frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1)-\lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1)\\
&=\sum_{k=0}^{k_1} \underline{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)}-\boxed{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)}+\frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1)-\underline{\lambda_1(k)F^{-1}(k)}\right]\tilde{\theta}(k+1)\\
&=\sum_{k=0}^{k_1} \underline{\frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)}-\boxed{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)}\\
&\quad+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1)
\end{aligned} \tag{4}$$

The term $\frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)$ is always none-negative if $1-\lambda_1(k)\geq 0$, which is the assumption in the forgetting factor definition. We only need to worry about

$$\sum_{k=0}^{k_1}\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1) \tag{5}$$

Appendix: derivation of (3)

Notice that

$$\frac{1}{2}\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]=\underline{\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)}+\frac{1}{2}\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)$$

In (5), there is a cross product term $\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)$ but no $\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)$. Add and substract terms to complete the squares. (5) becomes

$$\begin{aligned}
&\sum_{k=0}^{k_1}\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\frac{1}{2}\underline{\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)}\\
&+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1)-\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)+\underline{\frac{1}{2}\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)}\\
=&\sum_{k=0}^{k_1}\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1)-\frac{1}{2}\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)\\
&+\underbrace{\frac{1}{2}\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]}_{\geq 0}
\end{aligned}$$

Appendix: derivation of (3)

Summarizing, we get

$$\begin{aligned}
\sum_{k=0}^{k_1}w(k)s(k)=&\sum_{k=0}^{k_1}\frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)\\
&+\sum_{k=0}^{k_1}\frac{1}{2}\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]\\
&+\underbrace{\sum_{k=0}^{k_1}\frac{1}{2}\left[\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)\right]}_{\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}(k_1+1)\tilde{\theta}(k_1+1)-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)}
\end{aligned}$$

hence

$$\sum_{k=0}^{k_1}w(k)s(k)\geq-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)$$

Summary

1. Big picture
   Series-parallel adjustable system (equation-error method)
   Parallel adjustable system (output-error method)

2. RLS-based parallel PAA
   Formulas
   Stability requirement for PAAs with fixed adaptation gain
   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)