Diophantine Equation

Coprimeness of transfer functions

Theorem

Consider $G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n}$. $\alpha(z)$ *and* $\beta(z)$ *are coprime (no common roots) iff* S *(Sylvester matrix) is nonsingular:*

$$S = \begin{bmatrix} 1 & 0 & \dots & 0 & \beta_1 & 0 & \dots & \dots & 0 \\ \alpha_1 & 1 & \ddots & \vdots & \beta_2 & \beta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \alpha_1 & 1 & \beta_{n-1} & & \ddots & \ddots & 0 \\ \alpha_{n-1} & & & \alpha_1 & \beta_n & \ddots & & \ddots & \beta_1 \\ \alpha_n & \ddots & & \vdots & 0 & \beta_n & \ddots & & \beta_2 \\ 0 & \alpha_n & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \vdots & & \ddots & \beta_n & \beta_{n-1} \\ 0 & \dots & 0 & \alpha_n & 0 & \dots & \dots & 0 & \beta_n \end{bmatrix}_{(2n-1)\times(2n-1)}$$

Coprimeness of transfer functions

Example:

$$G(z) = \frac{z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-1}}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + 0}$$

i.e.

$$\beta_1 = 1$$
$$\beta_i = \alpha_{i-1} \ \forall i \geq 2$$
$$\alpha_n = 0$$

$\alpha(z)$ and $\beta(z)$ are not coprime, and S is clearly singular.

Solution concepts of Diophantine Equation

Theorem (Diophantine equation)

Given

$$\eta\left(z^{-1}\right) = 1 + \eta_1 z^{-1} + \eta_2 z^{-2} + \cdots + \eta_q z^{-q}$$
$$\alpha\left(z^{-1}\right) = 1 + \alpha_1 z^{-1} + \cdots + \alpha_n z^{-n}$$
$$\beta\left(z^{-1}\right) = \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_n z^{-n}$$

The Diophantine equation

$$\alpha\left(z^{-1}\right)\sigma\left(z^{-1}\right) + \beta\left(z^{-1}\right)\gamma\left(z^{-1}\right) = \eta\left(z^{-1}\right)$$

can be solved uniquely for $\sigma\left(z^{-1}\right)$ and $\gamma\left(z^{-1}\right)$

$$\sigma\left(z^{-1}\right) = 1 + \sigma_1 z^{-1} + \cdots + \sigma_{n-1} z^{-(n-1)}$$
$$\gamma\left(z^{-1}\right) = \gamma_0 + \gamma_1 z^{-1} + \cdots + \gamma_{n-1} z^{-(n-1)}$$

if the numerators of $\alpha\left(z^{-1}\right)$ and $\beta\left(z^{-1}\right)$ are coprime and $\deg\left(\eta\left(z^{-1}\right)\right) = q \leq 2n - 1$

Solution concepts of Diophantine Equation

Proof of Diophantine equation Theorem (key ideas):

$$\alpha\left(z^{-1}\right)\underbrace{\sigma\left(z^{-1}\right)}_{\text{unknown}}+\beta\left(z^{-1}\right)\underbrace{\gamma\left(z^{-1}\right)}_{\text{unknown}}=\eta\left(z^{-1}\right)$$

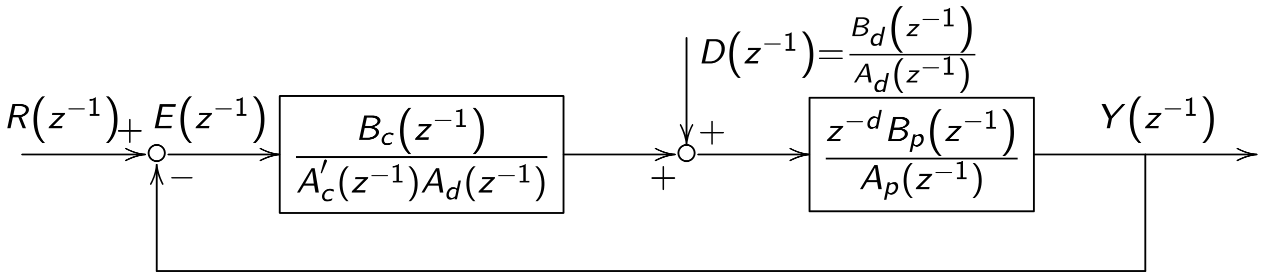
Matching the coefficients of z^{-i} gives

$$S\begin{bmatrix}\sigma_1\\\sigma_2\\\vdots\\\sigma_{n-1}\\\gamma_0\\\vdots\\\gamma_{n-1}\end{bmatrix}+\begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\\0\\\vdots\\0\end{bmatrix}=\begin{bmatrix}\eta_1\\\eta_2\\\vdots\\\eta_{n-1}\\\eta_n\\\vdots\\\eta_{2n-1}\end{bmatrix}$$

The coprime condition assures S is invertible. $\deg\eta\left(z^{-1}\right)\leq 2n-1$ assures the proper dimension on the right hand side of the equality.

Example

Application: Pole placement



Pole placement assigns the closed-loop characteristic equation:

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right)+A_p\left(z^{-1}\right)A_c'\left(z^{-1}\right)A_d\left(z^{-1}\right) = \underbrace{1+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}}_{\eta(z^{-1})}$$

which is in the structure of a *Diophantine equation*.
Design procedure: specify the desired closed-loop dynamics $\eta\left(z^{-1}\right)$; match coefficients of z^{-i} on both sides to get $B_c\left(z^{-1}\right)$ and $A_c'\left(z^{-1}\right)$.

Application: Pole placement

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right)+A_p\left(z^{-1}\right)A_c'\left(z^{-1}\right)A_d\left(z^{-1}\right) = \underbrace{1+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}}_{\eta(z^{-1})}$$

Questions:

- what are the constraints for choosing $\eta\left(z^{-1}\right)$?
  — depends on desired performance. Refer to undergraduate course on linear systems for concepts related to rise time, peak time, dampling ratio, etc.
- how to guarantee a unique solution of the controller?
  — addressed by solution concepts of Diophantine Eq.